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An Exactly Solvable Model for Algebraic Spin Liquid

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Abstract: For the term paper we worked on understanding the paper titled 'Algebraic Spin Liquid in an Exactly Solvable Spin Model' by Hong Yao, Shou-Cheng Zhang, and Steven A. Kivelson [1]. The paper introduces a new *exactly solvable* half-integer spin model on a square lattice whose ground state hosts an algebraic spin liquid. Our main focus was to understand the solution of the spin model described in the paper. Our report presents a self-contained (for an undergraduate student at our level) treatment of the properties of the spin model and its solution. The solution of the model relies on techniques developed by Alexei Kitaev while solving the his spin-1/2 model on a honeycomb lattice [2]. The learnt a variety of techniques prevalent in condensed matter research like Majorana and Dirac fermionisation, construction of gamma matrix models and basic lattice gauge theory. We also learnt about two new collective excitations in visons and spinons which are present in this model. In the end we worked with a symbolic manipulation software, SageMath, to diagonalise the Hamiltonian. In our report we discuss all of the above ideas in a (self-contained way) with the aim of obtaining the spectra of the Gamma Matrix Mdel (GMM).

Contents

1 Introduction

In this section we present a brief overview of what are we going to do in the following pages. The term paper that we selected for our project is slightly different from the most in the sense that we are not really looking at a physical system or trying to model a physical system directly. We rather look at a spin-3/2 system on a square lattice, with the Hamiltonian written in terms of four dimensional representations of the Clifford algebra, the gamma matrices. We do this to present an exactly solvable model for an algebraic spin liquid. Now this algebraic spin liquid is a phase of matter that is experimentally observed and the motivation to study this phase is that, it is this version of quantum spin liquid that we find in many materials. Having an exactly solvable model allows us to probe the properties of this phase in a toy model that may or may not be realisable. Because of this nature of this paper in this section we introduce the phases of matter that have motivated the construction of this model.

A Mott insulator is a phase of matter that defies conventional band theory. It is an insulating phase even though the Fermi level is within the band. A simple and intuitive

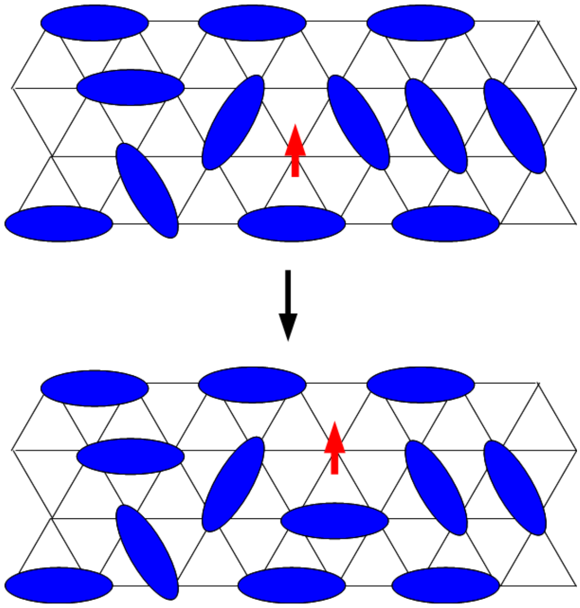


Figure 1. A schematic diagram of a quantum spin liquid on a triangular lattice. [3]

way of understanding a Mott insulator is through the Hubbard model. Consider a spin-1/2 system on a square lattice governed by the following Hamiltonian

$$H = -\sum_{ij,\sigma} t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma} + U \sum_{i} n_i(n_i - 1) \tag{1.1}$$

this is the Hubbard model, where σ is the spin index and n_i is the number operator on the site i. The creation and annihilation operators obey the canonical anti-commutation relations and t is the hopping parameter and U is the Coulomb energy-cost for occupying the same site. Notice that for very large values of U/t on average there is only one electron per site. Whereas for small values of U/t the electrons are completely delocalised and the model describes a metal. Now as U/t is increased from zero, after a critical critical value of U/t the model describes an insulator. A quantum phase transition has happened. This is a quantum metal to insulator transition (MIT). The insulator obtained is called a Mott insulator. They exist above the critical value of the U/t parameter of the model.

A quantum spin liquid is a phase of matter in which local spins are strongly correlated but they still fluctuate even at very low temperatures (which might as well be absolute zero). They are used to give a more sharp notion to the idea of a Mott insulator. Let us try to understand what a spin liquid is and its behavior in an intuitive way. We will take the example of a frustrated spin system as these systems provide a very simple and natural way of demonstrating spin fluctuations.

Consider spin-1/2 particles on a triangular lattice with an interaction of the form $\vec{S(i)} \cdot \vec{S(j)}$ note that this operator attains its minimum value when the spins are anti-aligned. So in the ground state we should have anti-ferromagnetic alignment. Except on a triangular lattice that is not possible. All the three spins cannot be anti-aligned with respect to each other. This forces the spins to be in an eternal state of fluctuations where they are sort of trying to anti-align but eventually cannot. Eventually the system forms bonds which are just the superposition of up-down and down-up configuration of spins on neighbouring sites and these bonds live along the lines joining the lattice sites. The bonds are shown in blue color in the figure. If we excite one of the spins as shown in red color, it can move across the lattice rearranging the bonds locally, such an excitation is called a spinon. So we see that a quantum spin liquid is phase of matter that is characterised by its short ranged spin correlation. However it should be noted that there are other phases of matter that have short ranged spin correlation that are not quantum spin liquid. Also there are Mott insulators that are not quantum spin liquids at low temperatures.

When the spectrum of the spinon excitation is gapless the phase is known as algebraic spin liquid. The motivation to study these phases stem from their possible observation in many materials. In this report we discuss one of the ways of studying this exotic quantum phase of matter, through an exactly solvable spin model. The structure of the report is as follows, we have already discussed the motivation for constructing an exactly solvable model for an algebraic spin liquid. In section 2, we will describe our system and write down its Hamiltonian in terms of Gamma matrices and discuss the symmetries of the model. Then in section 3, we begin setting up the solution of the model, we first implement a Majorana fermionisation procedure followed by a Dirac fermionisation, in between we discuss the ways of constraining the extended Hilbert space of the system obtained after Majorana fermionisation, we discuss the configuration of the ground state sector of the Dirac fermionised Hamiltonian and claim that our model has a ground state that is a quantum spin liquid. Once we know the ground state configuration and we have the Dirac fermionised Hamiltonian in section 4, we diagonalise it and obtain the spectrum of the system, upon analysing the spectrum we figure out the regime where the spectrum is gapless and hence hosts an algebraic spin liquid. In the final section we summarise our solution to give a bird's eye view of the whole solution.

2 Gamma Matrix Model

The Gamma matrix model (GMM) is defined on a square lattice with a spin-3/2 on each site. The model Hamiltonian is

$$\mathcal{H} = \sum_i \left(J_x \Gamma_i^1 \Gamma_{i+\hat{x}}^2 + J_y \Gamma_i^3 \Gamma_{i+\hat{y}}^4 + J_x' \Gamma_i^{15} \Gamma_{i+\hat{x}}^{25} + J_y' \Gamma_i^{35} \Gamma_{i+\hat{y}}^{45} - J_5 \Gamma_i^5 \right) \tag{2.1}$$

where the Γ matrices are 4×4 representation of the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$. This representation is obtained by symmetric bilinear combinations of the components of a spin-3/2 operator S^α. The Γ matrices are written explicitly in terms of S^α as follows

$$\Gamma^1 = \frac{1}{\sqrt{3}}\{S^y, S^z\} \qquad \Gamma^2 = \frac{1}{\sqrt{3}}\{S^z, S^x\} \qquad \Gamma^3 = \frac{1}{\sqrt{3}}\{S^x, S^y\} \tag{2.2}$$

$$\Gamma^4 = \frac{1}{\sqrt{3}}[(S^x)^2 - (S^y)^2] \qquad \Gamma^5 = (S^z)^2 - \frac{5}{4}. \tag{2.3}$$

Also, $\Gamma^{ab} = [\Gamma^a, \Gamma^b]/(2\iota)$ and i in the subscript of the Γ matrices labels the lattice site at $r_i = (x_i, y_i)$. The total number of spins on the site is $N = L_x L_y$ where L_x and L_y are the

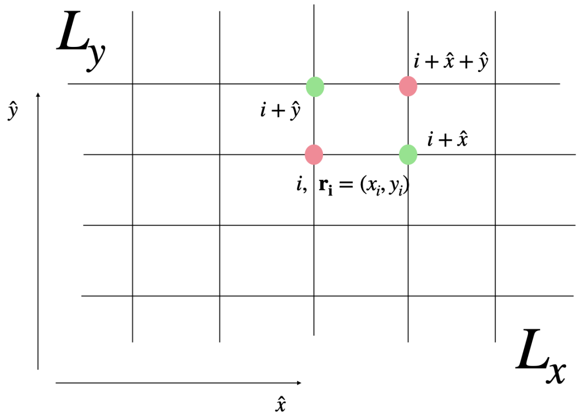


Figure 2. A schematic of the model governed by the Hamiltonian in 2.1.

linear sizes of the lattice, which we will assume to be even. We will also impose periodic boundary conditions.

2.1 Symmetries of the model

A few symmetries of the model are immediately clear from looking at the Hamiltonian.

1. The model respects translational symmetry. This is true because we have employed periodic boundary conditions, changing i by $i + n\hat{x}$ where $n \in \mathbb{Z}$ does not change the Hamiltonian, because all the terms will be recovered once the sum is performed.

2. The model respects time-reversal symmetry (TRS). The action of TR operator is flipping of spins. In the present Hamiltonian all the terms are either quadratic or quartic in spin operators and hence upon spin-flip operation the negative sign cancels out, leaving the Hamiltonian invariant.

3. GMM does not have SU(2) symmetry. Note that

$$[S^{\alpha}, \mathcal{H}] \neq 0 \tag{2.4}$$

because each term contain all the components of the spin-3/2 operator. Which implies spin at each site changes in the dynamics of the Hamiltonian and hence the Hamiltonian does not have SU(2) symmetry.

4. GMM does not even have U(1) symmetry. If the all the spins are rotated by the same amount the components of the spin-3/2 operators change and consequently, $\mathcal{H}$ may change.

5. Global Ising symmetry is present. If all the spins are rotated by π degrees about the z-axis, the effective transformation on the Hamiltonian is changing the sign of S^x and S^y, which renders the Hamiltonian invariant because $\mathcal{H}$ is quadratic in S^{β}, $\beta = x, y$.

3 Solving the GMM

In this section we attempt to solve the GMM using the ideas similar to the ones used by Kitaev in solving his honeycomb model. We define a plaquette operator $\hat{W}_i$ on plaquette i as follows

$$\hat{W}_i = \Gamma_i^{13}\Gamma_{i+\hat{x}}^{23}\Gamma_{i+\hat{y}}^{14}\Gamma_{i+\hat{x}+\hat{y}}^{24}. \tag{3.1}$$

We immediately note that

$$[\hat{W}_i, \hat{W}_j] = 0, \tag{3.2}$$

this happens because plaquette operators are composed of components of spin-3/2 operators and these components on two different lattice sites commute among themselves. A remarkable feature of the model that makes it solvable is an infinite set of conserved *fluxes*,

$$[\hat{W}_i, \mathcal{H}] = 0, \tag{3.3}$$

this is also readily seen, $\mathcal{H}$ has only one term with contribution only from $i-th$ site. Terms with contribution other than that from the $i-th$ site commute with $\hat{W}_i$ due to the reason mentioned above. Now the only term on the site i contains Γ^5, which is absent from the definition of $\hat{W}_i$ and hence all the Γ matrices used to define the plaquette operator commute with Γ^5 as evident from the Clifford algebra. So the plaquette operator commutes with the model Hamiltonian. The reason for calling the conservation of plaquette operator as conservation of *flux* will be clear in the next sections where we upon fermionisation we will see the emergence of $\mathbb{Z}_2$ gauge fields. Depending upon the values of these $\mathbb{Z}_2$ fields the eigenvalues of the Hamiltonian will change.

3.1 Fermionisation Procedure

We fermionise the model either in terms of Dirac fermions or Majorana fermions. The spin-3/2 operators can be expressed as bilinear forms of three flavours of Dirac fermions. The operators are explicitly

$$S^z = a^\dagger a + 2b^\dagger b - \frac{3}{2} \tag{3.4}$$

$$S^x + \iota S^y = S^+ = \sqrt{3}f^\dagger a + \sqrt{3}af + 2a^\dagger b. \tag{3.5}$$

This fermionisation procedure is subject to the constraint that the physical states have odd fermion parity, i.e., $(-1)^{\hat{N}} = -1$, where

$$\hat{N} = f^\dagger f + a^\dagger a + b^\dagger b \tag{3.6}$$

is the number operator for the Dirac fermions (it gives the total number of Dirac fermions of all the three flavours on each site). Instead of using this representation we will use the Majorana fermion representation. It involves representing a spin-3/2 particle in terms of six Majorana fermions. The Gamma matrices in terms of the six Majorana fermions are given by

$$\Gamma_i^\mu = \iota c_i^\mu d_i \qquad \Gamma_i^{\mu 5} = \iota c_i^\mu d_i' \qquad \Gamma_i^5 = \iota d_i d_i', \tag{3.7}$$

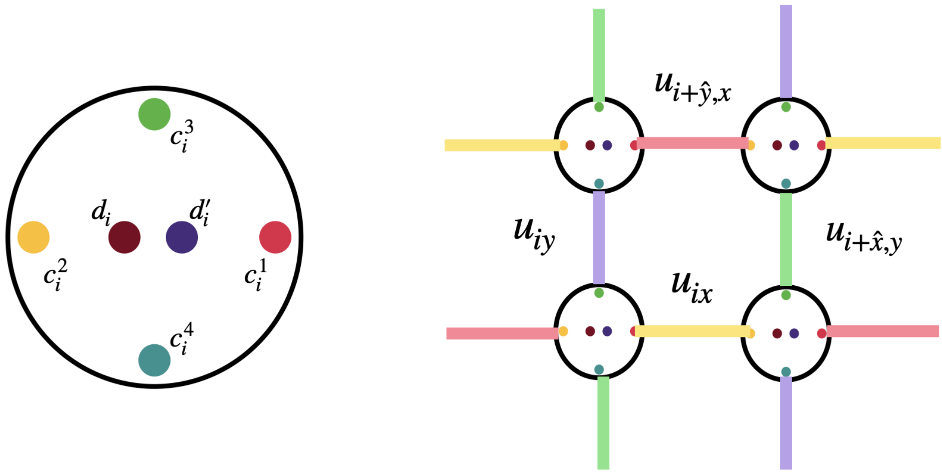


Figure 3. The transformation of each site upon Majorana fermionisation (right) and a representative plaquette of the whole system (left).

where c_i^μ, $\mu = 1, 2, 3, 4$, d_i and d_i' are the six Majorana fermions on the site i. The Majorana fermions form an eight dimensional Hilbert space. Which is enlarged from the four-dimensional Hilbert space of an spin-3/2. To find the physical states we will need a constraint on the states obtained from the fermionised Hamiltonian. Note that in terms of spin-3/2 operators we have

$$\Gamma_i^1 \Gamma_i^2 \Gamma_i^3 \Gamma_i^4 \Gamma_i^5 = -1, \tag{3.8}$$

for all the sites i. As a consequence we impose the following constraint on the physical states $|\Psi\rangle$ in terms of the Majorana fermions

$$D_i |\Psi\rangle = [-\iota c_i^1 c_i^2 c_i^3 c_i^4 d_i d_i'] |\Psi\rangle = |\Psi\rangle \,, \tag{3.9}$$

in a later section we will construct a projection operator that will project the states in the eight-dimensional Hilbert space of Majorana fermions to the physical subspace where the above constraint is met.

3.2 The Majorana fermion Hamiltonian

In this section we make some remarks on the Hamiltonian written in terms of Majorana fermions. We obtain the Hamiltonian by substituting the definition of Γ matrices in terms of Majorana fermions (3.7) in the model Hamiltonian (2.1) written in terms of Γ matrices,

$$\mathcal{H} = \sum_i \left(J_x \hat{u}_{ix} \iota d_i d_{i+\hat{x}} + J_y \hat{u}_{iy} \iota d_i d_{i+\hat{y}} + J_x' \hat{u}_{ix} \iota d_i' d_{i+\hat{x}}' + J_y' \hat{u}_{iy} \iota d_i' d_{i+\hat{y}}' - J_5 \iota d_i d_i' \right) \tag{3.10}$$

where $\hat{u}_{ix} = -\iota c_i^1 c_{i+\hat{x}}^2$ and $\hat{u}_{iy} = -\iota c_i^3 c_{i+\hat{y}}^4$. Note that,

$$[\hat{u}_{ix}, \mathcal{H}] = 0 \tag{3.11}$$

$$[\hat{u}_{iy}, \mathcal{H}] = 0. \tag{3.12}$$

This follows from the fact that,

$$[\hat{u}_{ix}, \hat{u}_{iy}] = 0 \tag{3.13}$$

as they contain different flavours of Majorana fermions. (3.11) implies that $\hat{u}_{i\lambda}$, $\lambda = x, y$ is a conserved quantity. We also have,

$$(\hat{u}_{i\lambda})^2 = 1 \tag{3.14}$$

because $(c_i^\mu)^2 = (d_i)^2 = (d_i')^2 = 0$, this is in accordance with second quantisation as well as the Clifford algebra followed by the Γ matrices. So the eigenvalues of $\hat{u}_{i\lambda}$, $u_{i\lambda} = \pm 1$. As a consequence of this the enlarged Hilbert space of Majorana fermions can be divided into sectors of definite $u_{i\lambda}$. In each sector $\{u\}$ the Hamiltonian will describe free Majorana fermions of c-flavour and interacting Majorana fermions of d-flavour.

$$\mathcal{H}(\{u\}) = \sum_i \left(J_x u_{ix} \iota d_i d_{i+\hat{x}} + J_y u_{iy} \iota d_i d_{i+\hat{y}} + J_x' u_{ix} \iota d_i' d_{i+\hat{x}}' + J_y' u_{iy} \iota d_i' d_{i+\hat{y}}' - J_5 \iota d_i d_i' \right) \tag{3.15}$$

Here, the $u_{i\lambda}$'s are the emergent $\mathbb{Z}_2$ gauge fields. The $\mathbb{Z}_2$ gauge transformations are given by

$$d_i \longrightarrow \Lambda_i d_i \tag{3.16}$$

$$d_i' \longrightarrow \Lambda_i d_i' \tag{3.17}$$

$$u_{i\lambda} \longrightarrow \Lambda_i u_{i\lambda} \Lambda_{i+\lambda} \tag{3.18}$$

where $\Lambda = \pm 1$. The identification of $\hat{u}_{i\lambda}$ as a $\mathbb{Z}_2$ gauge field is an artifact of its conservation and eigenvalues being ± 1.

The eigenstates of (3.10) live in the enlarged Hilbert space, these eigenstates can be written as the direct product of eigenstates of (3.15) ($|\Psi\rangle_{d,d'}$) and the c^μ-type Majorana fermion states ($|\Psi\rangle_c$).

$$|\Psi\rangle = |\Psi\rangle_c \otimes |\Psi\rangle_{d,d'} \tag{3.19}$$

The c^μ fermion state is defined to be the eigenstates of the gauge field operator

$$\hat{u}_{i\lambda} |\Psi\rangle_c = u_{i\lambda} |\Psi\rangle_c . \tag{3.20}$$

3.3 Degeneracy in the enlarged Hilbert space

The spectrum of $\mathcal{H}(\{u\})$ depends only on the following three gauge invariant quantities.

1. The flux on local plaquettes.

$$e^{(\iota\phi_i)} = u_{ix} u_{i+\hat{x}y} u_{iy} u_{i+\hat{y}x} \tag{3.21}$$

2. And two global fluxes,

$$e^{\iota\phi_x} = \prod_{i(y_i=1)} u_{ix}, \tag{3.22}$$

$$e^{\iota\phi_y} = \prod_{i(x_i=1)} u_{iy}. \tag{3.23}$$

Note that since $u_{i\lambda} = \pm 1$, we have ϕ_i and $\phi_\lambda = 0, \pi$. We also have,

$$\sum_i \phi_i = 0\ (mod\ 2\pi). \tag{3.24}$$

We demonstrate the correctness of (3.24) by considering different cases.

1. In case all the gauge fields take the value $+1$ or -1, all ϕ_i's are identically zero, and hence the above equality follows.

2. In case all the gauge fields but one has the same value, the odd gauge field will appear on two of the local fluxes and the sum of fluxes over the lattice would be 2π. This argument can be generalised to show that any gauge field would contribute to two local fluxes and hence π-fluxes will always occur in pair.

3. In case the gauge fields have values such that all the fluxes are π, the sum is $N\pi$, where N is the total number of lattice sites, which we have assumed to be even, once again the equality holds.

As a consequence of (3.24), if we know the flux through $N-1$ sites, the flux through the $N-th$ site is fixed. So there are only $N-1$ independent local fluxes. Along with the two global fluxes we get a total of $N+1$ independent fluxes. The total number of possible flux sectors is thus 2^{N+1}. Once we restrict ourselves to a specific flux sector $\{\phi\}$, because we have $2N$ $\mathbb{Z}_2$ gauge fields (two per site), the number of number of different gauge field choices corresponding to a unique flux sector is

$$\frac{\text{Gauge field combinations corresponding to each flux sector}}{\text{Total number of possible fluxes}} = \frac{2^{2N}}{2^{N+1}} = 2^{N-1}.$$

Since the spectrum only depends on the fluxes, these 2^{N-1} different choices lead to degenerate eigenstates. Hence, in the enlarged Hilbert space the degeneracy of each state is 2^{N-1}.

3.4 Projection to physical states

Most of the states in the enlarged Hilbert space of the fermionised Hamiltonian are not physical, i.e., they do not satisfy (3.9). The physical states form a subspace of the enlarged vector space. To obtain these physical states given an arbitrary state we construct a projection operator, that projects any state out of the physical subspace to the physical subspace, while it leaves states in the physical subspace invariant. This is realised through the following projection operator

$$|\Phi\rangle = \hat{P}\,|\Psi\rangle = \prod_i \left[\frac{1+D_i}{2}\right]|\Psi\rangle\,. \tag{3.25}$$

Where $|\Psi\rangle$ is the unphysical state and $|\Phi\rangle$ is the physical state. The working of projection operator is illustrated in the following remarks.

1. For a physical state $|\Psi\rangle$, we have, $D_i\,|\Psi\rangle = |\Psi\rangle$ from (3.9). Consequently, $\hat{P}\,|\Psi\rangle = |\Psi\rangle$. Hence, the physical state is left invariant by the projection operator.

2. For an unphysical state $|\Psi\rangle$, $D_i\,|\Psi\rangle = -1$. Consequently, $\hat{P}\,|\Psi\rangle = 0$. So unphysical states are annihilated by the projection operator. To understand why $D_i\,|\Psi\rangle = -1$, note that

   $$[D_i, \mathcal{H}] = 0, \tag{3.26}$$

   which implies D_i is conserved and,

   $$D_i^2 = 1 \tag{3.27}$$

   which implies the eigenvalues of D_i is ± 1, now the states with eigenvalue $+1$ are physical so the unphysical states are those with eigenvalue -1.

The explicit expression for the projection operator is

$$\hat{P} = \frac{\left[1 + \sum_i D_i + \sum_{i_1<i_2} D_{i_1} D_{i_2} + \cdots + \prod_i D_i\right]}{2^N}. \tag{3.28}$$

Note that D_i acting on an eigenstate of (3.10) is equivalent to a gauge transformation on the site i. So each of the terms in the explicit expression for $\hat{P}$ is equivalent to a gauge transformation on some subset of lattice sites. There is a small subtlety here, we know that there are only 2^{N-1} in-equivalent gauge transformations, but the sum in (3.28) contains 2^N terms. We make sense of the subtlety in following way. We begin by noting that $D = \prod_i D_i$ is a gauge transformation on each lattice site, thus it leaves all the gauge fields invariant. So we factor $\hat{P}$ in two terms, one containing D and the other containing all in-equivalent gauge transformations (P'),

$$\hat{P} = P' \frac{(1+D)}{2}. \tag{3.29}$$

We can do this factorisation and have $\hat{P}$ do the same job as before because D shares the same properties as D_i, as it is a product of D_i's, namely the following properties hold for D,

$$[D, \mathcal{H}] = 0 \tag{3.30}$$

$$D^2 = 1 \tag{3.31}$$

$$D\,|\Psi\rangle = \pm\,|\Psi\rangle\,. \tag{3.32}$$

So re-writing $\hat{P}$ in (3.28) has helped us appreciate the subtlety. The part of $\hat{P}$ which contains all the in-equivalent gauge transformations is responsible for mapping a physical state to the equal weight linear superposition of all in-equivalent gauge transformations acting on the physical state. Now there is another class of states that are annihilated by $\hat{P}$.

To classify states that are annihilated by $\hat{P}$, we fermionise the two d-flavour Majorana fermions at each site to one f-flavour Dirac fermion at each site. We define the Dirac fermions as follows

$$f_j = \iota^j \frac{d_j + \iota d'_j}{2}. \tag{3.33}$$

We can write D explicitly in terms of gauge fields and d-flavour Majorana fermions as

$$D = \prod_i [\hat{u}_{ix}\hat{u}_{iy}] \prod_i [\iota d_i d'_i]. \tag{3.34}$$

The first factor

$$\prod_i [\hat{u}_{ix}\hat{u}_{iy}] = (-1)^{\hat{N}_\phi} \tag{3.35}$$

where $\hat{N}_\phi$ is the number of π-fluxes through one of the two sublattices. The two sublattices refer to the two subset of sites that interact among themselves but not with each other. The product in (3.35) contains half the total fluxes on local plaquettes. These half of the fluxes correspond to all the fluxes on one of the sublattices. Now the choice of sublattice is not important because the total number of π-fluxes is even. We only count the number of π-fluxes because the contribution due to 0-fluxes is identity so it does not affect the operator D.

The second factor

$$\prod_i [\iota d_i d'_i] = (-1)^{\hat{N}_f} \tag{3.36}$$

where $\hat{N}_f = \sum_i f_i^\dagger f_i$ is number operator for the Dirac fermions. Note that because we have even number of lattice sites, the number of Dirac fermions is conserved (mod2). We write the d-flavour Majorana fermions in terms of Dirac fermions using (3.33)

$$d_i = \frac{f_j + (-1)^j f_j^\dagger}{\iota^j} \tag{3.37}$$

$$d'_i = \frac{f_j - (-1)^j f_j^\dagger}{\iota^{j+1}} \tag{3.38}$$

substituting this in (3.36) and noting that

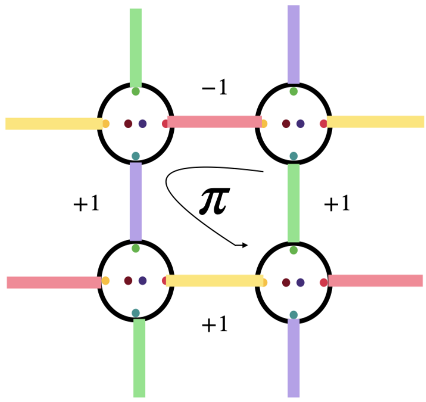
$$\{f_i, f_i\} = 0 \tag{3.39}$$

$$\{f_i^\dagger, f_i^\dagger\} = 0 \tag{3.40}$$

$$\{f_i^\dagger, f_i\} = 1 \tag{3.41}$$

we obtain $\prod_i [\iota d_i d'_i] = (-1)^{\hat{N}_f}$. As the terms containing only the creation and only the annihilation operators vanish. Both (3.36) and (3.35) together imply that

$$D = (-1)^{\hat{N}_\phi + \hat{N}_f}. \tag{3.42}$$



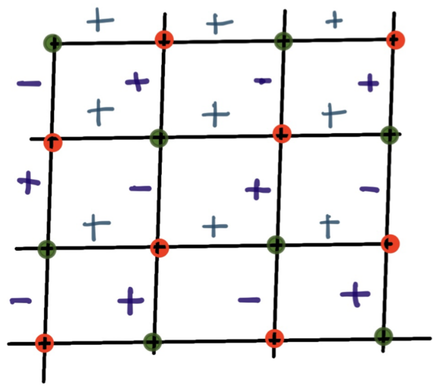


Figure 4. A representative plaquette with π-flux and a section of the lattice in π-flux ground state.

This relation give us a lot of insights on physical states. For example, consider the case where there is π-flux through each plaquette, in this case $\hat{N}_\phi$ is even and hence physical states are those with even Dirac fermionic parity. In contrast in case an odd number of π-fluxes are present on the sublattice, the physical states have odd number of Dirac fermions. Note that since D is conserved the fermion number parity is conserved, this tells us that the fermionic excitations are created by non-local operators.

3.5 π-flux states and gapped visons

In a given flux sector $\{\phi\}$ the lowest energy of the Hamiltonian $E_o(\{\phi\})$ can be obtained minimizing the energy $E_o(\{\phi\})$ with respect to $\{\phi\}$. The formal approach to this is done by deriving an effective action for the whole system by integrating over all fermions. But deriving it for such large system is quite non-trivial. Hence, for our system, one of Lieb's theorem comes to rescue.

3.5.1 Lieb's theorem and ground state

Before directly jumping into the theorem, let us first understand what does a bipartite lattice means. A bipartite lattice is one which can be expressed as sum of two sub lattices such that each site of a given sub lattice is connected only to the sites of other sub lattice. Lieb's Theorem states that the energy minimizing flux sector of a half-filled band of electron hopping on a planar bipartite lattice is the π-*flux sector*. Where a half filled band is one which we get when the no. of electron is half the number of available states. For example, consider the case when we have a lattice with N sites and each site contributes one electron. If we consider the the spin of electron then the no. of available states in the Brillouin zone

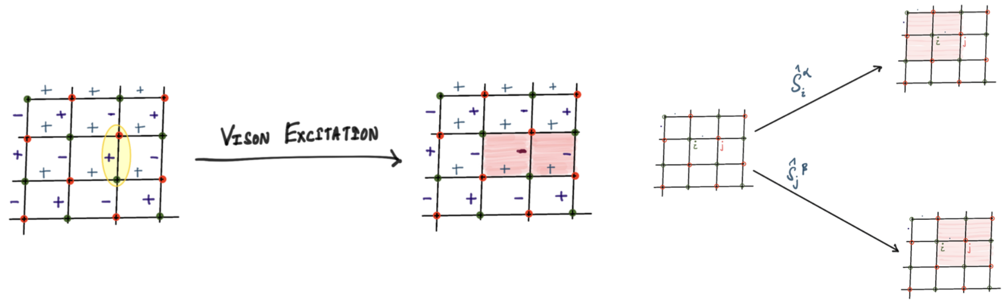


Figure 5. Vison excitations and the demonstration of creation of visons as an action of spin operators.

are $2N$ but the number of available electrons are only N and hence out states are half filled.

Coming to our case, the system we are studying, consists two Majorana fermions, d and d' hopping on a bipartite square lattice with a interaction term (governed by J_5). Each site contributes one type of Majorana fermion. Now, If we set $J_5 = 0$, then our system Hamiltonian comprises of only hopping terms and then we can apply Lieb's theorem to our system.. Using, Lieb's theorem we conclude that the ground state for our system is the π-flux sector i.e. all the plaquettes have π flux.

Now, we define, $\mathbb{Z}_2$ vortex excitations, 'visions' as the plaquettes with $\phi_i = 0$. This will happen if for a given square plaquette we get all four $u_{i\lambda}$ gauge fields to have same value or $+1$ and -1 in pairs. A vortex can be visualised as shown in figure below. To understand a vortex excitation let us consider the π-flux sector with the following gauge fields, $u_{ix} = 1$ and $u_{iy} = (-1)^i$. Now, if you take one u_{iy} and change it from -1 to 1 then we get two visons as shown in the figure.

Now, due to the constraint $\Sigma_i \phi_i = 0 \pmod{2\pi}$, we will always get visions in pairs. Now, as $[u_{i\lambda}, \mathcal{H}] = 0$ the visons we get are non-dynamical in nature. The minimum energy required to get a vision excitation is finite an given by

$$\Delta_\nu \sim \left(\sqrt{|J_x J_y|} + \sqrt{|J'_x J'_y|} \right). \tag{3.43}$$

Since the visions are gapped, if we slightly change J_5 following the condition, $0 < J_5 << \Delta_\nu$, we will still remain in the π-flux sector. Now, for rest of the report we will focus our study on spinon (spin excitation) in the π-flux sector regime.

3.6 Spin correlations

The ground state i.e. the π-flux sector in which we have uniform π-flux, has translational symmetry because shifting the whole lattice by each plaquette leaves the whole flux distribution across the lattice invariant. The above state also shows time-reversal symmetry. Time reversal operator performs following transformation, $u_{i\lambda} \rightarrow u_{i\lambda}$. On performing these transformation the overall flux through each plaquette remains invariant and hence the state has time-reversal symmetry.

To show that the system is a spin liquid we need to show that the spin-spin correlation are short ranged and hence there is no magnetic ordering. As described above, the spin-3/2 operator can be written as bilinear forms of Majorana operators, for example, $S_i^z = ic_i^3 c_i^4 + ic_i^1 c_i^2$. Therefore, the spin operator S_i^α either creates or annihilates bond-fermions (c_i fermions) at site i and hence the the gauge field undergoes following transformation $u_{i\lambda} \rightarrow -u_{i\lambda}$. As a result, the flux through neighbouring four plaquettes changes. Now, if we go on calculating the spin-spin correlation given as $\langle\psi| S_i^\alpha S_j^\beta |\psi\rangle$, we get it to be non-zero for neighbouring i and j sites and zero for all other cases, because only in the neighbouring case the gauge field returns to the same value and hence a non-zero inner product of the states. Such a nature of correlation function shows that indeed spin-spin interaction is short ranged and hence our model is a spin-liquid.

4 Gapless fermions

To obtain the excitation spectrum in the π-flux sector, which is also the ground state sector, we fix the gauge by choosing $u_{ix} = 1$, $u_{iy} = (-1)^i$. This ensures that the fluxes on all the local plaquette is π. The Hamiltonian in terms of free Dirac fermions is obtained by substituting (3.37) in (3.15).

$$H_0 = \sum_i \left(t_x f_u^\dagger f_{i+\hat{x}} + t_y (-1)^i f_i^\dagger f_{i+\hat{y}} - J_5 f_i^\dagger f_i - \Delta_x (-1)^i f_i^\dagger f_{i+\hat{x}}^\dagger - \Delta_y f_i^\dagger f_{i+\hat{y}}^\dagger + \text{h.c.} \right) \quad (4.1)$$

where $t_\lambda = J_\lambda + J'_\lambda$ and $\Delta_\lambda = J_\lambda - J'_\lambda$. This is a tight-binding nearest neighbour interaction Hamiltonian for Dirac fermions. This describes a p-wave superconductor of spinless fermions.

4.1 Spinon excitation spectrum

We go to the momentum space to get the spectrum as a function of the wavevector. We do this by writing the Hamiltonian in its Fourier space using the definition of Bloch states,

$$f_{\mathbf{k}} = \sum_i \exp(-\iota \mathbf{k} \cdot \mathbf{r_i}) \frac{f_i}{\sqrt{N}}. \quad (4.2)$$

The Hamiltonian in the Fourier space is given by

$$H_0 = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}^\dagger H_{\mathbf{k}} \Phi_{\mathbf{k}} \quad (4.3)$$

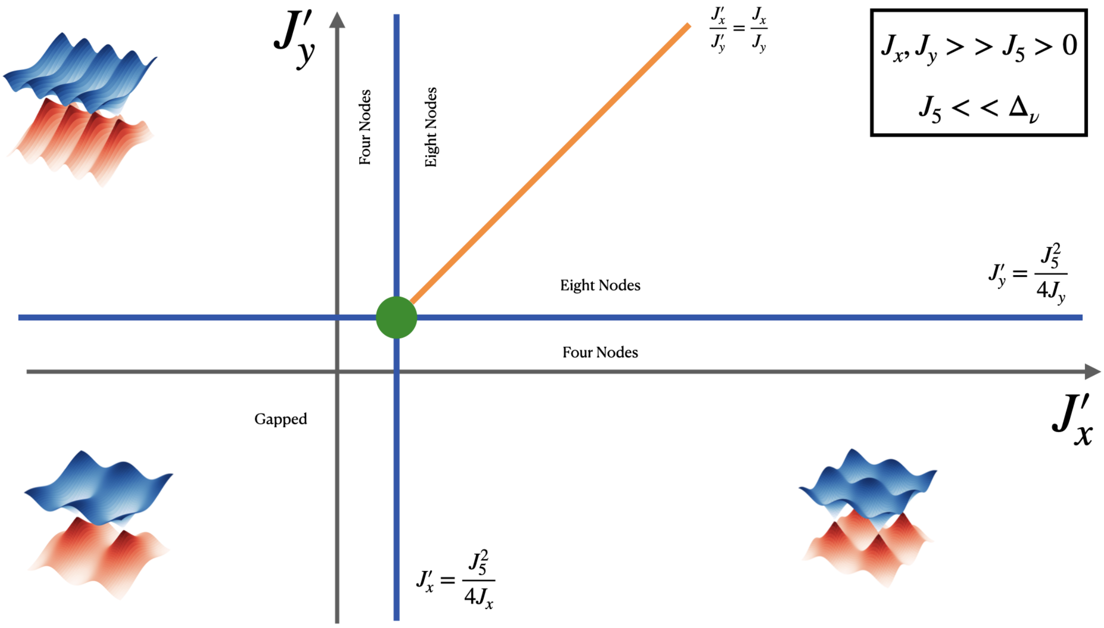


Figure 6. The quantum phase diagram of the GMM in the uniform π-flux ground state sector. (Note that the band diagrams are not exact but representative.)

where $\Phi^\dagger_{\mathbf{k}} = \left(f^\dagger_{\mathbf{k}}, f^\dagger_{\mathbf{k}+\mathbf{Q}}, f^\dagger_{-\mathbf{k}}, f^\dagger_{-\mathbf{k}-\mathbf{Q}}\right)$, is the Bloch basis vector. Note that $\mathbf{Q} = (\pi, \pi)$ is the reciprocal lattice vector, and due to the translational symmetry of the model, because of which $\mathbf{k}$ is equivalent to $\mathbf{k}+\mathbf{Q}$, the sum in (4.3) is performed over half the Brillouin zone. The model also have time reversal symmetry, i.e., $H_{\mathbf{k}} = H_{-\mathbf{k}}$. The translational and the time reversal symmetry allows us to easily compute the matrix form of $H_{\mathbf{k}}$.

$$H = \begin{bmatrix} 2t_x \cos k_x - 2J_5 & 2\imath t_y \sin k_y & 2\imath \Delta_y \sin k_y & 2\Delta_x \cos k_x \\ -2\imath t_y \sin k_y & -2t_x \cos k_x - 2J_5 & -2\Delta_x \cos k_x & -2\imath \Delta_y \sin k_y \\ -2\imath \Delta_y \sin k_y & -2\Delta_x \cos k_x & -2t_x \cos k_x + 2J_5 & -2\imath \Delta_y \sin k_y \\ 2\Delta_x \cos k_x & 2\imath \Delta_y \sin k_y & 2\imath t_y \sin k_y & 2t_x \cos k_x + 2J_5 \end{bmatrix} \tag{4.4}$$

The eigenvalues of the matrix gives the quasiparticle spinon excitation spectrum.

$$E_{\pm,\mathbf{k}} = 2\sqrt{J_5^2 + 2g_{+,\mathbf{k}} \pm 2\sqrt{g_{-,\mathbf{k}}^2 + J_5^2 g_{\mathbf{k}}}} \tag{4.5}$$

Where,

$$g_{\pm,\mathbf{k}} = (J_x^2 \pm J_x'^2)\cos^2 k_x + (J_y^2 \pm J_y'^2)\sin^2 k_y, \tag{4.6}$$

$$g_{\mathbf{k}} = (J_x + J'_x)^2 \cos^2 k_x + (J_y^2 + J'_y)^2 \sin^2 k_y. \tag{4.7}$$

In the following sections we study the model in different regimes of the interaction parameter strengths.

4.2 The $\mathbf{J_5 = 0}$ regime

In this case we have,

$$E_{+,\mathbf{k}} = 4\left(J_x^2 \cos^2 kx + J_y^2 \sin^2 k_y\right)^{1/2}, \tag{4.8}$$

$$E_{-,\mathbf{k}} = 4\left(J_x'^2 \cos^2 kx + J_y'^2 \sin^2 k_y\right)^{1/2}. \tag{4.9}$$

Both the spectra is gapless. Around the Dirac points the spectrum is linear in momentum, so the spinon excitation are massless Dirac fermions.

4.3 The $\mathbf{0 < J_5 << \Delta_v}$ regime

This is the regime where the J_5 is non-zero but very small compared to the vison gap energy Δ_v. In this regime the ground state lies in the π-flux sector as discussed above. The features of the spectrum in this regime is as follows

1. $E_{+,\mathbf{k}}$ is always gapped as $J_5 > 0$.
2. $E_{-,\mathbf{k}}$ may be gapped or gapless depending on the values of J_x, J_y and J_x', J_y'. The condition for gapless excitations are,

$$J_x J_x' \cos^2 k_x + J_y J_y' \sin^2 k_y = \frac{J_5^2}{4}, \tag{4.10}$$

$$\left(J_x J_x' - J_y J_y'\right) \cos k_x \sin k_y = 0. \tag{4.11}$$

3. In the regime $J_x, J_y >> J_5 > 0$, where $J_5 << \Delta_v$ is satisfied for arbitrary J_x' and J_y'. We have the following features in the phase diagram of the model as a function of J_x' and J_y'.
   (a) When

$$J_x' > \frac{J_5^2}{4J_x}, \qquad J_y' > \frac{J_5^2}{4J_y}, \qquad \frac{J_x'}{J_y'} \neq \frac{J_x}{J_y},$$

   the fermion spectrum has eight Dirac nodes.

   (b) When

$$J_x' > \frac{J_5^2}{4J_x}, \qquad J_y' < \frac{J_5^2}{4J_y},$$

   there are four Dirac nodes.

   (c) When

$$J_x' < \frac{J_5^2}{4J_x}, \qquad J_y' > \frac{J_5^2}{4J_y},$$

   once again there are four Dirac nodes but at different locations than in the previous case.

(d) When

$$J'_x < \frac{J_5^2}{4J_x}, \qquad\qquad J'_y < \frac{J_5^2}{4J_y},$$

there are no Dirac nodes, all the spinon excitations are gapped in this case. In this regime, the gapless spinon excitations, if they exist are stable in the sense that any weak translational and time reversal invariant perturbation only shifts the positions of the nodes but not its topology. Consequently the present phase with Dirac nodes is characteristic of a stable quantum phase of matter, i.e., the current phase is an algebraic spin liquid.

5 Conclusion

Now that we have obtained the spectrum and the energy phase diagram of the model it is a good time to take a look back and summarise what we did, why did we need the projection operator, why care about gauge fields, why not write the Dirac Hamiltonian first up! All of these questions would hopefully be answered in the following concluding (summarising) remarks.

1. First of all we wrote the Hamiltonian for the GMM using four dimensional representations of the Clifford algebra. A natural question to ask, after all the analysis upon which we realise that the model that is actually easy to solve is the Majorana fermionised one, is why not start with the Majorana fermionised model and solve it, why bother with the GMM? The Majorana fermionised model also has the added advantage of having a clear physical interpretation in terms of nearest neighbour Majorana fermion-fermion interaction. The answer lies in the fact that construction of Majorana fermionised model would be difficult given the enormous number of combinations the nearest neighbour interactions may be present in the model owing to the presence of large number of Majorana fermions at each lattice site. Also while making spin models one of the first constraining arguments is that of symmetries of the system and these symmetries are more evident and easy to enforce in spin models.

2. Once we wrote the model Hamiltonian the symmetries of the system became evident immediately. Apart from the traditional symmetries like translational and Ising the model also an infinite set of conserved fluxes. It is this infinite set of conserved quantity that renders the model solvable. We exploited these symmetries in finding the flux sector of the ground state. This also shows why writing the spin Hamiltonian gives more insights about the system. We know how to solve spin models with an infinite set of conserved fluxes from Kitaev's analysis, one would like their model to have these symmetries and it is easy to include them while writing the Hamiltonian in terms of spin operators.

3. This infinitely many conserved fluxes motivate us to attempt to use Kitaev's approach to solve our system. And so we fermionise the model using six Majorana fermions.

We have to use six Majorana fermions because we have a spin-3/2 system. The fermionisation procedure leads to the extension of the Hilbert space of the system. In the new extended Hilbert space we notice one nice and one rather disturbing feature.

(a) The nice feature is that among the six Majorana fermions four are present in the fermionised model with no interaction among themselves of with the other two. While the remaining two Majorana fermions have nearest neighbour interactions.

(b) And the annoying feature is that a large number of degeneracy is present in the enlarged physical space. This is annoying because all our (perceived) progress seems to be useless. The problem remains as difficult as in the beginning.

4. The problem of large number of degeneracy is solved by a smart construction of an operator that acts on states in this enlarged Hilbert space. The action of this projection operator is to project unphysical states to physical states. It does so by constructing a linear combination of unphysical states such that the combination satisfies our definition of physical state.

5. We find that the projection operator obtains a rather simple form when we Dirac fermionise the Majorana fermionised Hamiltonian. The structure of the Hamiltonian also gives us many constraints on the kind of states that we can observe. These constraints manifest as fermionic parity and kind of fluxes present on the lattice plaquettes. To obtain the ground state sector, we state a theorem by Lieb. We do not get into the details of Lieb's theorem because of the nature of his original paper that uses ideas from graph theory. We briefly discuss the configuration of the ground state from various considerations and motivate it physically. We also see that the spin correlations for the model are short ranged. In fact they are only nearest neighbour. We could have computed this quantity numerically for various interaction parameter by due to the requirement of computational resources could not do so.

6. Once we had all the above constraints on the ground state. We finally write the Hamiltonian in its matrix form and attempt to diagonalise it in k-space. We analyse the obtained spectrum in the regime where the spinonic excitations are present but the vison excitations are not present. We already know at this point that the ground state is a quantum spin liquid due to short-ranged spin-correlation. Upon further analysis we obtain a region where the spinon excitations are gapless. This is what we set out to achieve, such a state is what we call an algebraic spin liquid.

The GMM has other nice properties that we did not have the time and skills to look into, these include the behaviour along the critical line in the phase diagram and the behaviour of the system in the presence of vison excitations.

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